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Power quasi Sujatha Distribution with properties and applications in real lifetime data

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Abstract

This study presents a three-parameter power quasi Sujatha distribution. Statistical properties including the survival function, hazard function, reverse hazard function, mean residual life function and stochastic ordering have been discussed. Moments of the proposed distribution have been obtained. The estimation of the parameters using the maximum likelihood method and maximum product spacing estimation has been explained and a simulation study has been presented to determine the efficiency of the maximum likelihood estimate of the parameters. The bootstrap confidence interval method has been used to estimate the confidence interval of the parameters. Finally, two examples of real lifetime datasets have been presented to demonstrate the applications of the proposed distribution. Also, the goodness of fit test shows a better fit compared to the three-parameter power Sujatha distribution, power quasi Lindley distribution, generalized gamma distribution, three-parameter Sujatha distribution and three-parameter generalized Lindley distribution.

Key words: quasi Sujatha distribution, statistical properties, maximum likelihood estimation, maximum product spacing estimation, applications.

1. Introduction

A one-parameter lifetime distribution known as the Lindley distribution was first developed by Lindley (1958) using the convex combination of the gamma distribution and exponential distribution. Ghitany et al. (2008) subsequently studied statistical characteristics and goodness of fit of the Lindley distribution and showed that Lindley provides better fit as compared to exponential distribution. Using the convex combination approach, Shanker (2016a) proposed a one-parameter Sujatha distribution (SD). It offers better fit on some datasets than the exponential and Lindley distributions. The probability density function (pdf) and the cumulative density function (cdf) of SD are given by

$$f(x;\eta) = \frac{\eta^3}{\eta^2+\eta+2}(1+x+x^2)e^{-\eta x}\ ;\ x>0, \eta>0 \qquad (1)$$

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$$F(x;\eta) = 1 - \left[1 + \frac{\eta x(\eta x+\eta+2)}{\eta^2+\eta+2}\right] e^{-\eta x}\ ;\ x > 0, \eta > 0 \tag{2}$$

The statistical properties of SD have been discussed by Shanker (2016a). Although SD provides a better fit than exponential and Lindley distributions, it has been noted that because exponential, Lindley and SD have just one parameter, they do not give adequate fits for some dataset. A two-parameter quasi Sujatha distribution (QSD) was proposed by Shanker (2016b) by adding an additional parameter in the pdf of SD, which has more flexibility as compared to SD. QSD is defined by its pdf and cdf as

$$f(x;\eta,\omega) = \frac{\eta^2}{\omega\eta+\eta+2}(\omega + \eta x + \eta x^2)e^{-\eta x}; x > 0, \eta > 0, \omega > 0 \tag{3}$$

$$F(x;\eta) = 1 - \left[1 + \frac{\eta x(\eta x+\eta+2)}{\eta\omega+\eta+2}\right] e^{-\eta x}\ ;\ x > 0, \eta > 0\ , \omega > 0 \tag{4}$$

Various researchers have proposed several power versions of the lifetime distribution using the power transformation $X = Y^{\frac{1}{\beta}}$. For instance, Weibull (1951) introduced Weibull distribution (WD) from exponential distribution, Ghitany et al. (2013) introduced power Lindley distribution (PLD) from Lindley distribution of Lindley (1958), Shanker and Shukla (2017) introduced power Shanker distribution from Shanker (2015), Shukla (2019) introduced power Pranav distribution from Pranav distribution of Shukla (2018), Aderoju and Adeniyi (2022) introduced power generalized Akash distribution (PGAD) from generalized Akash distribution of Shanker et al. (2018), Shanker and Shukla (2018) introduced power Aradhana distribution from Aradhana distribution of Shanker (2016c), Power Sujatha distribution (PSD) was proposed by Shanker and Shukla (2019) from SD of Shanker (2016a), Prodhani and Shanker (2024) introduced power Pratibha distribution (PPD) from Pratibha distribution of Shanker (2023), Alkarni (2015) proposed the power quasi Lindley distribution (PQLD) from quasi Lindley distribution (QLD) of Shanker and Mishra's (2013). Two particular cases of PQLD are the Lindley distribution and the power Lindley distribution provided by Ghitany et al. (2013). Stacy (1962) proposed generalized gamma distribution (GGD) from the gamma distribution, Prodhani and Shanker (2024) introduced three-parameter power Sujatha distribution (TPPSD) from two-parameter Sujatha distribution (TPSD) of Mussie and Shanker (2018). Recently, Nwikpe and Iwok (2020) proposed a three-parameter Sujatha distribution (ATPSD), and later its various statistical properties and applications were studied by Prodhani and Shanker (2023). Nosakhare and Festus (2018) proposed three-parameter generalized Lindley distribution (TPGLD).

Adding an additional parameter on QSD using the power transformation technique offers substantially greater flexibility with a pdf capable of representing unimodal, bimodal and heavy-tailed data along with a wide range of skewness and kurtosis. Its hazard function may demonstrate non-decreasing and non-increasing or non-monotonic behaviors. In contrast, the QSD is restricted to unimodal, positively skewed shapes with an exclusively non-decreasing hazard rate and thus limiting its applicability to more complex lifetime data. The primary reasons for considering the power quasi Sujatha distribution (PQSD) are:

i. Its pdf shows unimodal, bimodal, heavy-tailed and a wide range of skewness and kurtosis patterns. Its hazard function is non-increasing and non-decreasing.
ii. Unlike gamma distribution, the cdf and survival function of PQSD come in a closed form.
iii. The proposed model retains mathematical tractability and includes SD, QSD and PSD as particular cases.

In Section 2, the pdf and cdf of PQSD along with graphical representation of the pdf are presented. In Section 3, moments of PQSD are presented. In Section 4, reliability properties including the survival function, hazard function, mean residual life function, reverse hazard function and stochastic ordering are discussed. In Section 5, maximum likelihood estimation, Fisher's information matrix, maximum product spacing estimation and Bootstrap confidence interval are discussed. In Section 6, a simulation study has been conducted the using acceptance-rejection method of simulation to examine the consistency of the estimator. In Section 7, a goodness of fit measures are demonstrated on two real lifetime datasets. In Section 8, the conclusion of the study is presented.

2. Power quasi Sujatha distribution

Considering the power transformation $X = Y^{\frac{1}{\tau}}$ in the pdf of QSD, the pdf of PQSD can be obtained as

$$f(x;\eta,\omega,\tau) = \frac{\tau\eta^2}{\omega\eta+\eta+2}(\omega+\eta x^{\tau}+\eta x^{2\tau})x^{\tau-1}e^{-\eta x^{\tau}}; x>0, \omega>0, \eta>0, \tau>0 \quad (5)$$

$$= p_1 f_1(x;\eta,\tau) + p_2 f_2(x;\eta,\tau) + (1-p_1-p_2) f_3(x;\eta,\tau)$$

where $p_1 = \frac{\omega\eta}{\omega\eta+\eta+2}$, $p_2 = \frac{\eta}{\omega\eta+\eta+2}$, $f_1(x;\eta,\tau) = \tau\eta x^{\tau-1}e^{-\eta x^{\tau}}$,

$f_2(x;\eta,\tau) = \frac{\tau\eta^2}{\Gamma(2)}x^{2\tau-1}e^{-\eta x^{\tau}}$, and $f_3(x;\eta,\tau) = \frac{\tau\eta^3}{\Gamma(3)}x^{3\tau-1}e^{-\eta x^{\tau}}$

This means that PQSD is a convex combination of WD(η,τ), GGD$(2,\eta,\tau)$and GGD$(3,\eta,\tau)$. The three particular cases of PQSD are SD, PSD and QSD for particular values of the parameters$\tau = 1, \omega = \eta$; $\omega = \eta$ and$\tau = 1$, respectively. The corresponding cdf of PQSD can be obtained as

$$F(x;\eta,\omega,\tau) = 1 - \left[1 + \frac{\eta x^{\tau}(\eta x^{\tau}+\eta+2)}{\omega\eta+\eta+2}\right]e^{-\eta x^{\tau}}; x>0, \omega>0, \eta>0, \tau>0 \quad (6)$$

From Figure 1, it is clear that for various values of the parameters, PQSD has unimodal, bimodal, positively skewed and heavy tailed nature and hence PQSD can be a suitable model for real lifetime data of unimodal, bimodal, positively skewed and heavy-tailed natures.

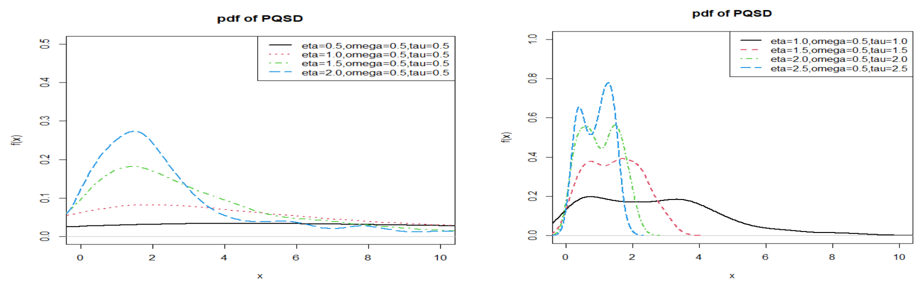

Figure1. pdf of PQSD

3. Moments and their related measures

The rth raw moment of PQSD can be obtained as

$$\mu_r' = E(X^r) = \frac{r\Gamma\left(\frac{r}{\tau}\right)}{\tau\eta^{\frac{r}{\tau}}(\omega\eta+\eta+2)}\left[\omega\eta + \frac{\eta(r+\tau)}{\tau} + \frac{(r+\tau)(r+2\tau)}{\tau^2}\right]$$

$$= \frac{r\Gamma\left(\frac{r}{\tau}\right)}{\tau^3\eta^{\frac{r}{\tau}}(\omega\eta+\eta+2)}\left[\omega\eta\tau^2 + \eta(r+\tau)\tau + (r+\tau)(r+2\tau)\right]; r = 1,2,3\ldots \quad (7)$$

Putting $r = 1,2,3,4$ in (7), we get the first four raw moments as

$$\mu_1' = \frac{\Gamma\left(\frac{1}{\tau}\right)}{\tau^3\eta^{\frac{1}{\tau}}(\omega\eta+\eta+2)}\left[\omega\eta\tau^2 + \eta(1+\tau)\tau + (1+\tau)(1+2\tau)\right]$$

$$\mu_2' = \frac{2\Gamma\left(\frac{2}{\tau}\right)}{\tau^3\eta^{\frac{2}{\tau}}(\omega\eta+\eta+2)}\left[\omega\eta\tau^2 + \eta(2+\tau)\tau + (2+\tau)(2+2\tau)\right]$$

$$\mu_3' = \frac{3\Gamma\left(\frac{3}{\tau}\right)}{\tau^3\eta^{\frac{3}{\tau}}(\omega\eta+\eta+2)}\left[\omega\eta\tau^2 + \eta(3+\tau)\tau + (3+\tau)(3+2\tau)\right]$$

$$\mu_4' = \frac{4\Gamma\left(\frac{4}{\tau}\right)}{\tau^3\eta^{\frac{4}{\tau}}(\omega\eta+\eta+2)}\left[\omega\eta\tau^2 + \eta(4+\tau)\tau + (4+\tau)(4+2\tau)\right]$$

The variance of PQSD can be obtained as

$$\mu_2 = \mu_2' - (\mu_1')^2$$

$$= \frac{2\tau^3(\omega\eta+\eta+2)\{4+2(3+\eta)\tau+\tau^2(\omega\eta+\eta+2)\}\Gamma\left(\frac{2}{\tau}\right) - \left[((\omega+1)\tau^2+\tau)\eta+2\tau^2+3\tau+1\right]^2\left(\Gamma\left(\frac{1}{\tau}\right)\right)^2}{\tau^6\eta^{\frac{2}{\tau}}(\omega\eta+\eta+2)^2}$$

The expressions for μ_3 and μ_4 are not given because their expressions are in disordered forms.

4. Reliability properties of PQSD

4.1. Survival function

The survival function of PQSD can be obtained as

$$S(x;\eta,\omega,\tau) = 1 - F(x;\eta,\omega,\tau) = \left[\frac{\eta x^{\tau}(\eta x^{\tau}+\eta+2)+(\omega\eta+\eta+2)}{(\omega\eta+\eta+2)}\right] e^{-\eta x^{\tau}} \tag{8}$$

4.2. Hazard function

The hazard function of PQSD can be obtained as

$$h(x;\eta,\omega,\tau) = \frac{\tau\eta^2(\omega+\eta x^{\tau}+\eta x^{2\tau})x^{\tau-1}}{\eta x^{\tau}(\eta x^{\tau}+\eta+2)+(\omega\eta+\eta+2)} \tag{9}$$

Figure 2 of the hazard function shows that for $\tau < 1$ and higher values of (η,ω), it has monotonically decreasing hazard suitable for modelling early failure or infant mortality scenarios and for $\tau \geq 1$ and higher values of (η,ω), it has monotonically increasing hazard appropriate for aging or wear-out processes.

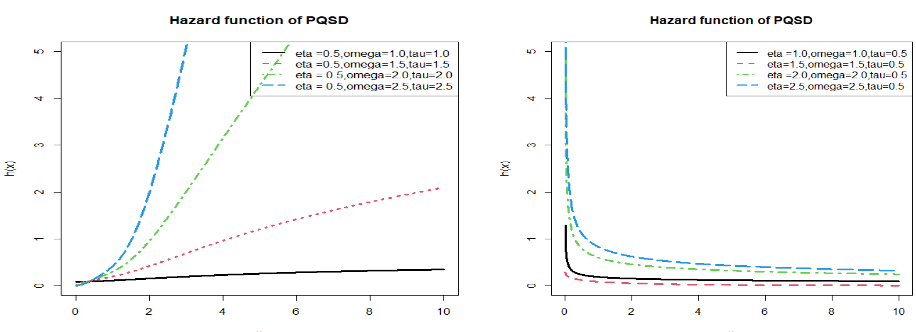

Figure 2. Hazard function of PQSD

4.3. Mean residual life function

The mean residual life function of PQSD can be obtained as

$$m(x;\eta,\omega,\tau) = E[X - x | X \geq x] = \frac{1}{1 - F(x;\eta,\omega,\tau)} \int_x^{\infty} [1 - F(t;\eta,\omega,\tau)]\, dt$$

$$= \frac{\eta\left[(\omega\eta+\eta+2)\Gamma\left(\frac{1}{\tau},\eta x^{\tau}\right)+(\eta+2)\Gamma\left(\frac{1}{\tau}+1,\eta x^{\tau}\right)+\Gamma\left(\frac{1}{\tau}+2,\eta x^{\tau}\right)\right]}{\tau\eta^{\frac{1}{\tau}}[\omega\eta+\eta+2+\eta x^{\tau}(\eta x^{\tau}+\eta+2)]e^{\eta x^{\tau}}} \tag{10}$$

The mean residual life function in Figure 3 shows that for any values of the parameters, it is consistently decreasing as time progresses.

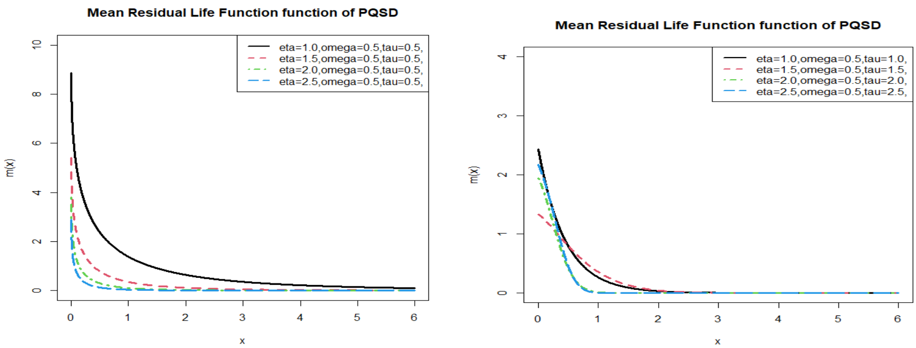

Figure 3. Mean residual life function of PQSD

4.4. Reverse hazard function

The reverse hazard function of PQSD can be obtained as

$$r(x;\eta,\omega,\tau)=\frac{\tau\eta^{2}(\omega+\eta x^{\tau}+\eta x^{2\tau})x^{\tau-1}e^{-\eta x^{\tau}}}{(\omega\eta+\eta+2)-[(\omega\eta+\eta+2)+\eta x^{\tau}(\eta x^{\tau}+\eta+2)]e^{-\eta x^{\tau}}};x>0,\omega>0,\eta>0,\tau>0 \tag{11}$$

The reverse hazard function in Figure 4 shows that for any values of the parameters, it is consistently decreasing as time progresses.

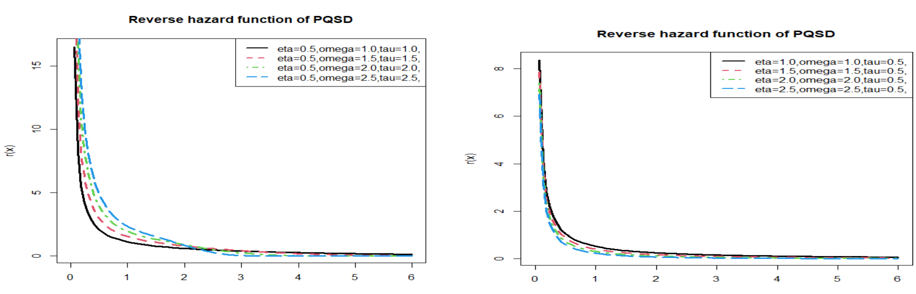

Figure 4. Reverse hazard function of PQSD

4.5. Stochastic ordering

Shaked and Shantikumar (1994) provided the following result for the stochastic ordering of distributions:

$$\begin{array}{c} X<_{lr}Y\Rightarrow X\underset{\substack{\Downarrow\\ X<_{st}Y}}{<_{hr}}Y\Rightarrow X<_{mrl}Y \end{array}$$

Theorem 1: Let $X\sim$ PQSD (η_1,ω_1,τ_1) and $Y\sim$ PQSD(η_2,ω_2,τ_2). If $\eta_1>\eta_2,\omega_1=\omega_2,\tau_1=\tau_2$ or $\tau_1<\tau_2,\omega_1=\omega_2,\eta_1=\eta_2$or $\omega_1>\omega_2,\eta_1=\eta_2,\tau_1=\tau_2$ then $X<_{lr}Y$, hence $X<_{hr}Y, X<_{mrl}Y$ and $X<_{st}Y$.

Proof: We have

$$\frac{f_X(x)}{f_Y(x)}=\frac{\tau_1{\eta_1}^2(\omega_2\eta_2+\eta_2+2)}{\tau_2{\eta_2}^2(\omega_1\eta_1+\eta_1+2)}\left(\frac{\omega_1+\eta_1x^{\tau_1}+\eta_1x^{2\tau_1}}{\omega_2+\eta_2x^{\tau_2}+\eta_2x^{2\tau_2}}\right)x^{\tau_1-\tau_2}e^{-(\eta_1x^{\tau_1}-\eta_2x^{\tau_2})}$$

$$log\left(\frac{f_X(x)}{f_Y(x)}\right)=log\left[\frac{\tau_1{\eta_1}^2(\omega_2\eta_2+\eta_2+2)}{\tau_2{\eta_2}^2(\omega_1\eta_1+\eta_1+2)}\right]+log\left[\frac{\omega_1+\eta_1x^{\tau_1}+\eta_1x^{2\tau_1}}{\omega_2+\eta_2x^{\tau_2}+\eta_2x^{2\tau_2}}\right]$$
$$+(\tau_1-\tau_2)\,log\,x-(\eta_1x^{\tau_1}-\eta_2x^{\tau_2})$$

$$\frac{d}{dx}\left[log\left(\frac{f_X(x)}{f_Y(x)}\right)\right]$$
$$=\frac{\eta_1\tau_1x^{\tau_1-1}+2\eta_1\tau_1x^{2\tau_1-1}}{\omega_1+\eta_1x^{\tau_1}+\eta_1x^{2\tau_1}}-\frac{\eta_2\tau_2x^{\tau_2-1}+2\eta_2\tau_2x^{2\tau_2-1}}{\omega_2+\eta_2x^{\tau_2}+\eta_2x^{2\tau_2}}$$
$$+\frac{\tau_1-\tau_2}{x}-(\eta_1\tau_1x^{\tau_1-1}-\eta_2\tau_2x^{\tau_2-1})$$

Thus, for $\eta_1>\eta_2,\omega_1=\omega_2,\tau_1=\tau_2$ or $\tau_1<\tau_2,\omega_1=\omega_2,\eta_1=\eta_2$ or $\omega_1>\omega_2,\eta_1=\eta_2,\tau_1=\tau_2$, $\frac{d}{dx}\left[log\left(\frac{f_X(x)}{f_Y(x)}\right)\right]<0$. This means that $X<_{lr}Y$, hence $X<_{hr}Y$, $X<_{mrl}Y$ and $X<_{st}Y$.

For example, when $\omega_1=1>\omega_2=0,\eta_1=\eta_2=1,\tau_1=\tau_2=1$

$$\frac{d}{dx}\left[log\left(\frac{f_X(x)}{f_Y(x)}\right)\right]=-\frac{(1+2x)}{(1+x+x^2)(x+x^2)}<0$$

Thus, $\frac{d}{dx}\left[log\left(\frac{f_X(x)}{f_Y(x)}\right)\right]<0$ for all $x>0$

5. Estimation of the parameters

5.1. Maximum likelihood estimation of the parameters

Let $(x_1,x_2,\ldots,x_n)$ represent a random sample from PQSD(η,ω,τ). The log-likelihood function of PQSD is given by

$log\,L=n[2\,log\,\eta+log\,\tau-log(\eta\omega+\eta+2)]+\sum_{i=1}^{n}log(\omega+\eta{x_i}^{\tau}+\eta{x_i}^{2\tau})+(\tau-1)\sum_{i=1}^{n}log\,x_i-\eta\sum_{i=1}^{n}{x_i}^{\tau}$. Now, the log-likelihood equations are given by

$$\frac{\partial\,log\,L}{\partial\eta}=\frac{2n}{\eta}-\frac{n(\omega+1)}{\omega\eta+\eta+2}+\sum_{i=1}^{n}\frac{{x_i}^{\tau}+{x_i}^{2\tau}}{\omega+\eta{x_i}^{\tau}+\eta{x_i}^{2\tau}}-\sum_{i=1}^{n}{x_i}^{\tau}=0$$

$$\frac{\partial \log L}{\partial \omega} = -\frac{n\eta}{\omega\eta + \eta + 2} + \sum_{i=1}^{n} \frac{1}{\omega + \eta x_i^{\tau} + \eta x_i^{2\tau}} = 0$$

$$\frac{\partial \log L}{\partial \tau} = \frac{n}{\tau} + \sum_{i=1}^{n} \frac{\eta x_i^{\tau} \log x_i + 2\eta x_i^{2\tau} \log x_i}{\omega + \eta x_i^{\tau} + \eta x_i^{2\tau}} + \sum_{i=1}^{n} \log x_i - \eta \sum_{i=1}^{n} x_i^{\tau} \log x_i = 0.$$

Due to the lack of closed-form expressions for these three log-likelihood equations, the likelihood function must be solved iteratively in R software using maximization techniques until sufficiently close parameter values are obtained.

For finding the MLEs $(\hat{\eta}, \hat{\omega}, \hat{\tau})$ of parameters (η, ω, τ) of PQSD, following equations can be solved:

$$\begin{bmatrix} \frac{\partial^2 \log L}{\partial \eta^2} & \frac{\partial^2 \log L}{\partial \eta \partial \omega} & \frac{\partial^2 \log L}{\partial \eta \partial \tau} \\ \frac{\partial^2 \log L}{\partial \omega \partial \eta} & \frac{\partial^2 \log L}{\partial \omega^2} & \frac{\partial^2 \log L}{\partial \omega \partial \tau} \\ \frac{\partial^2 \log L}{\partial \tau \partial \eta} & \frac{\partial^2 \log L}{\partial \tau \partial \omega} & \frac{\partial^2 \log L}{\partial \tau^2} \end{bmatrix}_{\substack{\hat{\eta}=\eta_0 \\ \hat{\omega}=\omega_0 \\ \hat{\tau}=\tau_0}} \begin{bmatrix} \hat{\eta} - \eta_0 \\ \hat{\omega} - \omega_0 \\ \hat{\tau} - \tau_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial \log L}{\partial \eta} \\ \frac{\partial \log L}{\partial \omega} \\ \frac{\partial \log L}{\partial \tau} \end{bmatrix}$$

where η_0, $\omega_0$and $\tau_0$are the preliminary values of η, ω and τ. These equations are resolved iteratively until close estimates of parameters are obtained.

Thus, Fisher's information matrix is obtained as

$$I = -E \begin{bmatrix} \frac{\partial^2 \log L}{\partial \eta^2} & \frac{\partial^2 \log L}{\partial \eta \partial \omega} & \frac{\partial^2 \log L}{\partial \eta \partial \tau} \\ \frac{\partial^2 \log L}{\partial \omega \partial \eta} & \frac{\partial^2 \log L}{\partial \omega^2} & \frac{\partial^2 \log L}{\partial \omega \partial \tau} \\ \frac{\partial^2 \log L}{\partial \tau \partial \eta} & \frac{\partial^2 \log L}{\partial \tau \partial \omega} & \frac{\partial^2 \log L}{\partial \tau^2} \end{bmatrix} = \begin{bmatrix} I_{\eta\eta} & I_{\eta\omega} & I_{\eta\tau} \\ I_{\omega\eta} & I_{\omega\omega} & I_{\omega\tau} \\ I_{\tau\eta} & I_{\tau\omega} & I_{\tau\tau} \end{bmatrix}$$

The solution of Fisher's information matrix will provide asymptotic variance and covariance of the maximum likelihood estimator for $(\hat{\eta}, \hat{\omega}, \hat{\tau})$. The approximate $100(1-\alpha)\%$ confidence intervals for (η, ω, τ) are $\hat{\eta} \pm Z_{\frac{\alpha}{2}} \sqrt{\frac{{I_{\eta\eta}}^{-1}}{n}}$, $\hat{\omega} \pm Z_{\frac{\alpha}{2}} \sqrt{\frac{{I_{\omega\omega}}^{-1}}{n}}$ and $\hat{\tau} \pm Z_{\frac{\alpha}{2}} \sqrt{\frac{{I_{\tau\tau}}^{-1}}{n}}$ respectively, where Z_{α} is the upper $100\alpha^{\text{th}}$ percentile of the standard normal distribution.

Theorem 2. MLEs of PQSD are consistent estimators. That is,
as $n \to \infty$, $P\{|\hat{\eta} - \eta| > \varepsilon\} \to 0$, $P\{|\hat{\omega} - \omega| > \varepsilon\} \to 0$,$P\{|\hat{\tau} - \tau| > \varepsilon\} \to 0$

Proof: The asymptotic variance of MLEs is given by
$V(\hat{\eta}, \hat{\omega}, \hat{\tau}) = \frac{1}{n} I^{-1}(\hat{\eta}, \hat{\omega}, \hat{\tau})$. Therefore, $V(\hat{\eta}, \hat{\omega}, \hat{\tau}) \to 0$ as $n \to \infty$

By Chebyshev's inequality, we have

$P\{|\hat{\eta}-\eta|>\varepsilon\} \le \frac{V(\hat{\eta})}{\varepsilon^2} = \frac{1}{n}\frac{I_{\eta\eta}{}^{-1}}{\varepsilon^2}$. As $n \to \infty$, $V(\hat{\eta}) = \frac{I_{\eta\eta}{}^{-1}}{n} \to 0$.

Thus, $P\{|\hat{\eta}-\eta|>\varepsilon\} \to 0$ as $n \to \infty$. Hence, $\hat{\eta} \xrightarrow{P} \eta$. Similarly, it can be shown that $\widehat{\omega} \xrightarrow{P} \omega$ and $\hat{\tau} \xrightarrow{P} \tau$.

5.2. Maximum Product Spacing Estimation

The maximum product spacing estimates (MPSE) $(\hat{\eta}, \widehat{\omega}, \hat{\tau})$ of parameters (η, ω, τ) of PQSD can be obtained numerically by maximizing the following function with respect toη, ω and τ.

$$MPSE = \frac{1}{n+1}\sum_{i=1}^{n+1} log[F(x_i, \eta, \omega, \tau) - F(x_{i-1}, \eta, \omega, \tau)].$$

5.3. Bootstrap Confidence Intervals

The bootstrap is a powerful, data-driven technique for assessing the sampling variability of estimates and constructing confidence intervals without relying on strong parametric assumptions (Efron and Tibshirani, 1993). Let $\widehat{H}$ [where $\widehat{H} = (\hat{\eta}, \widehat{\omega}, \hat{\tau})$] be the maximum likelihood estimate of a parameter H, [where $H = (\eta, \omega, \tau)$] based on an observed sample $x = (x_1, x_2, \dots, x_3)$. The percentile-bootstrap procedure proceeds as follows:

1. **Resampling**: Draw Mbootstrap samples $x_1{}^*, x_2{}^*, \dots, x_M{}^*$ using sampling with replacement from the original data x.
2. **Re-estimation**: Compute the estimate $\widehat{H}_m$for each bootstrap sample $x_m{}^*$.
3. **Bootstrap Distribution**: The set $\{H_1{}^*, H_2{}^*, \dots, H_M{}^*\}$ empirically approximates the sampling distribution of $\widehat{H}$.
4. **Percentile Interval**: For a nominal $100(1-\alpha)\%$interval, take$\frac{\alpha}{2}$and $1-\frac{\alpha}{2}$quantiles of the $\{\widehat{H}_m\}$as the confidence limits (Davison and Hinkley, 1997).

Because it does not assume normality of $\widehat{H}$ or rely on Fisher's information, the bootstrap is particularly useful for complex models, small samples, or when the likelihood surface is irregular.

6. A simulation study

To understand the flexibility and performance of the maximum likelihood estimators (MLEs) of PQSD, we carried out a simulation study. We looked at the variances, mean square errors (MSEs), biases (B), mean estimates and the approximate confidence intervals of MLEs. The findings are shown in Tables 1, 2, and 3. Using the formulas listed below, the mean, bias, MSE, and variance are computed.

$Mean = \frac{1}{n}\sum_{i=1}^{n}\widehat{H}_i$, $B = \frac{1}{n}\sum_{i=1}^{n}(\widehat{H}_i - H)$, $MSE = \frac{1}{n}\sum_{i=1}^{n}(\widehat{H}_i - H)^2$, $Variance = MSE - B^2$, where $H = (\eta, \omega, \tau)$ denotes the true parameter vector and $\widehat{H}_i = (\hat{\eta}_i, \widehat{\omega}_i, \hat{\tau}_i)$denotes the estimates of the parameter vector from the ith simulated sample.

The simulation used to carry out this work is run in the following manner:

a. The acceptance-rejection simulation technique is used to generate data. The steps in this method are as follows:
   i. Generating Y distributed as Gamma(η, ω)
   ii. Generating U distributed as Uniform$(0,1)$
   iii. If $U \leq \frac{f(y)}{Mg(y)}$, then set $X = Y$, (accept the sample and if not reject the sample), if rejected, go through steps (i–iii) again until the appropriate samples are obtained.
   Here $M$is a fixed number.
b. Each sample size is replicated 10000 times.

Means, Biases, MSEs, and variances of MLEs of the PQSD parameters decrease and the mean value of PQSD tends to the true parameter value with an increasing sample size, which is consistent with the first-order asymptotic theory of MLE.

Table 1. Descriptive constants of PQSD for $\eta = 3$,$\omega = 3$and $\tau = 2.7$

Parameters	Sample size	Mean	Bias	MSE	Variance	95% CI	
						Lower	Upper
η	20	3.02511	0.02511	0.00346	0.00283	3.00179	3.04842
	50	3.02023	0.02024	0.00323	0.00282	3.00551	3.03495
	100	3.01906	0.01906	0.00249	0.00213	3.01001	3.02810
	200	3.01597	0.01597	0.00223	0.00197	3.00981	3.02212
	300	3.01348	0.01348	0.00201	0.00183	3.00863	3.01832
ω	20	3.13014	0.13014	0.06463	0.04770	3.03442	3.22585
	50	3.06471	0.06471	0.04583	0.04164	3.00814	3.12127
	100	3.05477	0.05477	0.03658	0.03358	3.00890	3.10064
	200	3.04411	0.04411	0.02184	0.01989	3.02456	3.06365
	300	3.03385	0.03362	0.01485	0.01372	3.02059	3.04710
τ	20	2.72175	0.02175	0.00107	0.00060	2.71101	2.73248
	50	2.71661	0.01661	0.00087	0.00059	2.70987	2.72334
	100	2.71390	0.01390	0.00068	0.00048	2.70960	2.71819
	200	2.70844	0.00844	0.00051	0.00044	2.70553	2.70553
	300	2.70508	0.00508	0.00046	0.00043	2.70273	2.70742

Table 2. Descriptive constants of PQSD for $\eta = 2, \omega = 1.4$and $\tau = 1.7$

Parameters	Sample size (n)	Mean	Bias	MSE	Variance	95% CI	
						Lower	Upper
η	20	2.02125	0.02125	0.00344	0.00299	1.99728	2.04521
	50	2.01888	0.01888	0.00261	0.00225	2.00573	2.03202
	100	2.01690	0.01690	0.00213	0.00185	2.00847	2.02533
	200	2.01369	0.01369	0.00149	0.00131	2.00867	2.01870
	300	2.01159	0.01159	0.00124	0.00110	2.00783	2.01534
ω	20	1.35863	-0.04137	0.00964	0.00792	1.31962	1.39763
	50	1.36715	-0.03284	0.00862	0.00754	1.34308	1.39121
	100	1.36826	-0.03173	0.00806	0.00705	1.35180	1.38471
	200	1.38303	-0.01696	0.00695	0.00667	1.37171	1.39434
	300	1.38509	-0.01490	0.00579	0.00558	1.37663	1.39354
τ	20	1.66402	-0.03597	0.01060	0.00930	1.62175	1.70628
	50	1.67882	-0.02117	0.00464	0.00419	1.66087	1.69676
	100	1.68551	-0.01448	0.00273	0.00252	1.67567	1.69534
	200	1.68922	-0.01077	0.00156	0.00144	1.68396	1.69447
	300	1.69394	-0.00605	0.00121	0.00117	1.69006	1.69781

Table 3. Descriptive constants of PQSD for $\eta = 0.3, \omega = 0.3$,and $\tau = 0.3$

Parameters	Sample size (n)	Mean	Bias	MSE	Variance	95% CI	
						Lower	Upper
η	20	0.31240	0.01240	0.00350	0.00330	0.28722	0.33757
	50	0.30870	0.00870	0.00280	0.00270	0.29429	0.32310
	100	0.30590	0.00590	0.00210	0.00210	0.29691	0.31488
	200	0.30380	0.00380	0.00160	0.00160	0.29825	0.30934
	300	0.30210	0.00210	0.00130	0.00130	0.29801	0.30618
ω	20	0.32470	0.02470	0.06460	0.06400	0.21382	0.43557
	50	0.31520	0.01520	0.04580	0.04560	0.25600	0.37439
	100	0.30980	0.00980	0.03660	0.03650	0.27235	0.34724
	200	0.30640	0.00640	0.02180	0.02180	0.28593	0.32686
	300	0.30430	0.00430	0.01490	0.01480	0.29053	0.31806
τ	20	0.29530	-0.00470	0.00110	0.00100	0.28144	0.30915
	50	0.29760	-0.00240	0.00090	0.00090	0.28928	0.30591
	100	0.29890	-0.00110	0.00070	0.00070	0.29371	0.30408
	200	0.29960	-0.00040	0.00050	0.00050	0.29650	0.29650
	300	0.29980	-0.00020	0.00050	0.00050	0.29726	0.30233

7. Applications

The applications of PQSD have been evaluated using the two real-life datasets from the flood peaks of the Wheaton River and engineering and both datasets are over-dispersed shown by the data summary of both datasets in Table 4 as PQSD is suitable for over-dispersed data. The datasets are as follows.

Dataset 1: The following right-skewed data present the exceedances of flood peaks (in m3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. The data consist of 72 exceedances for the years 1958–1984, rounded to one decimal place. This data were analyzed by Choulakian and Stephens (2001) and are given as follows:

1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 1.9, 13, 12, 9.3, 1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1,1.1, 2.5, 14.4, 1.7, 37.6, 0.6, 2.2, 39, 0.3, 15, 11, 7.3, 22.9, 0.1, 1.7, 1.1, 0.6, 9, 1.7, 7, 20.1,0.4, 2.8, 14.1, 9.9, 10.4, 10.7, 30, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64, 1.5, 2.5, 27.4, 1, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5, 27.

Dataset 2: The following right-skewed data set discussed by Picciotto R. (1970) is used to correspond to the time-to-failure of a polyester/viscose yarn in a textile experiment for testing the tensile fatigue characteristics of yarn. It consists of a sample of 100 cm yarn at 2.3% strain level. The values are:

86, 146, 251, 653, 98, 249, 400, 292, 131, 169, 175, 176, 76, 264, 15, 364, 195, 262, 88, 264, 157, 220, 42, 321, 180, 198, 38, 20, 61, 121, 282, 224, 149, 180, 325, 250, 196, 90, 229, 166, 38, 337, 65, 151, 341, 40, 40, 135, 597, 246, 211, 180, 93, 315, 353, 571, 124, 279, 81, 186, 497, 182, 423, 185, 229, 400, 338, 290, 398, 71, 246, 185, 188, 568, 55, 55, 61, 244, 20, 284, 393, 396, 203, 829, 239, 236, 286, 194, 277, 143, 198, 264, 105, 203, 124, 137, 135, 350, 193, 188.

Table 4. Summary of the dataset 1 and 2

Datasets	Minimum	1st Quartile	Median	Mean	3rdQuartile	Maximum	Variance
1	0.100	2.125	9.500	12.204	20.125	64.000	151.221
2	15.000	129.200	195.500	222.000	282.500	829.00	20914.380

The parameter estimates of PQSD, TPPSD, PQLD, GGD, ATPSD and TPGLD along with their standard errors for datasets 1 and 2, obtained using the MLE and MPSE methods, are displayed in Tables 5 and 6. The values of $-2\,log\,L$, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC), Hannan-Quinn Information Criterion (HQIC), Kolmogorov-Smirnov (K-S) Statistics for the above two datasets have been computed and presented in Tables 7 and 8 using the formulas:

$AIC = -2\,log\,L + 2p,$

$BIC = -2\,log\,L + p\,log(n),$

$CAIC = -2\,log\,L + 2\frac{pn}{n-p-1}$,

$HQIC = -2\,log\,L + 2p\,log(log(n))$ K-S $= \underset{x}{Sup}|G_n(x) - G_0(x)|$, where p =the number of parameters, n =sample size, $G_n(x) =$ empirical cdf of the considered distribution and $G_0(x) =$ cdf of the considered distribution.

The Bootstrap confidence intervals for the dataset 1 and 2 are presented in Table 9.

Table 5. MLE and MPSE of the parameters for the dataset 1

Distributions	MLE of the dataset 1			MPSE of the dataset 1		
	$\hat{\eta}$ SE($\hat{\eta}$)	$\hat{\omega}$ SE($\hat{\omega}$)	$\hat{\tau}$ SE($\hat{\tau}$)	$\hat{\eta}$ SE($\hat{\eta}$)	$\hat{\omega}$ SE($\hat{\omega}$)	$\hat{\tau}$ SE($\hat{\tau}$)
PQSD	0.2050 (0.0794)	15.1360 (13.4909)	0.8862 (0.0978)	0.2332 (0.0944)	12.9389 (12.4735)	0.8366 (0.0985)
TPPSD	0.8085 (0.3134)	0.1000 (31.0460)	0.5338 (0.1852)	0.4005 (0.0001)	2.0959 (0.7919)	3.8690 (0.0001)
PQLD	0.1536 (0.0831)	2.5343 (4.2815)	0.8704 (0.1102)	0.1647 (0.0996)	4.8160 (15.3272)	0.7985 (0.0946)
GGD	0.0143 (0.0215)	0.5382 (0.2058)	1.3659 (0.3619)	0.1000 (0.0624)	0.9525 (0.2205)	0.9168 (0.1344)
ATPSD	0.1442 (0.0176)	0.0100 (0.0104)	0.0100 (…)	0.1635 (0.0195)	30.4630 (41.8227)	65.41357 (92.6840)
TPGLD	0.1536 0.0832	16.4826 35.8484	0.8704 0.1102	0.1758 (0.1070)	12.9611 (31.5808)	0.8210 (0.1174)

Table 6. MLE and MPSE of the parameters for the dataset 2

Distributions	MLE of the dataset 2			MPSE of the dataset 2		
	$\hat{\eta}$ SE($\hat{\eta}$)	$\hat{\omega}$ SE($\hat{\omega}$)	$\hat{\tau}$ SE($\hat{\tau}$)	$\hat{\eta}$ SE($\hat{\eta}$)	$\hat{\omega}$ SE($\hat{\omega}$)	$\hat{\tau}$ SE($\hat{\tau}$)
PQSD	0.0282 (0.0214)	0.2987 (6.3510)	0.8663 (0.1279)	0.0178 (0.0097)	10.0303 (14.5112)	0.9380 (0.0915)
TPPSD	0.0100 (0.0023)	0.3000 247.5289	1.0490 (0.0423)	0.0118 (0.0181)	1.9570 (79.3259)	0.9633 (0.2466)
PQLD	0.0255 (0.0102)	0.2429 (…)	0.8019 (0.0751)	0.0889 (0.0004)	2.2284 (0.0004)	0.8524 (0.0004)
GGD	0.0284 (0.0136)	2.9520 (0.4996)	0.8611 (0.0649)	0.0315 (0.0958)	2.8945 (2.9351)	1.5122 (0.7290)
ATPSD	0.0135 0.0007	31.0945 (…)	0.0168 (0.2214)	0.0135 (0.0008)	4.3605 (1.2517)	0.0100 (0.1974)
TPGLD	0.1000 (0.0174)	0.1000 (0.6290)	0.5822 (0.0372)	0.0070 (0.0021)	1.6096 (9.2574)	1.0398 (0.0510)

Table 7. Goodness of fit measures for dataset 1

Distributions	$-2\ log\ L$	AIC	BIC	CAIC	HQIC	MLE		MPSE	
						K-S	p-value	K-S	p-value
PQSD	**500.60**	**506.60**	**513.43**	**506.95**	**509.31**	**0.10**	**0.50**	**0.08**	**0.72**
TPPSD	505.10	511.10	517.93	511.45	513.82	0.13	0.22	0.72	0.00
PQLD	502.73	508.73	515.56	509.08	511.44	0.16	0.07	0.15	0.09
GGD	502.16	508.16	514.99	508.51	510.87	0.15	0.09	0.09	0.61
ATPSD	505.47	511.47	518.30	511.82	514.18	0.16	0.05	0.29	0.00
TPGLD	502.73	508.73	515.56	509.08	511.44	0.13	0.19	0.12	0.30

Table 8. Goodness of fit measures for dataset 2

Distributions	$-2\ log\ L$	AIC	BIC	CAIC	HQIC	MLE		MPSE	
						K-S	p-value	K-S	p-value
PQSD	**1251.10**	**1257.10**	**1264.91**	**1257.35**	**1260.26**	**0.11**	**0.26**	**0.06**	**0.90**
TPPSD	1559.25	1565.25	1573.06	1565.50	1568.41	0.14	0.06	0.11	0.36
PQLD	1274.31	1280.31	1288.12	1280.56	1283.47	0.22	0.00	0.79	0.00
GGD	1251.27	1257.27	1265.08	1257.52	1260.43	0.19	0.00	0.92	0.00
ATPSD	1255.82	1261.82	1269.63	1262.07	1264.98	0.18	0.00	0.07	0.69
TPGLD	1301.53	1307.53	1315.34	1307.78	1310.69	0.39	0.00	0.07	0.69

Table 9. Bootstrap confidence interval for dataset 1 and 2

Datasets	Parameters	95% CI	
		Lower	Upper
1	$\hat{\eta}$	0.1286	0.2863
	$\hat{\omega}$	8.2674	38.2731
	$\hat{\tau}$	0.7762	1.0430
2	$\hat{\eta}$	0.0074	0.0522
	$\hat{\omega}$	0.0010	41.6467
	$\hat{\tau}$	0.7642	1.0934

From Tables 7 and 8 it is obvious that PQSD provides better fit as compared to TPPSD, PQLD, GGD, PSD, ATPSD and TPGLD because PQSD has the least $-2\ log\ L$, AIC, BIC, CAIC, HQIC and K-S values. From the K-S values given by MLE and MPSE, MPSE is better than MLE in terms of the fit for both the datasets. Further, PQSD offers a far better fit than TPPSD, PQLD, GGD, ATPSD, TPGLD as shown by the fitted plot, quantile-quantile (Q-Q) plot, probability-probability (P-P) plot and empirical cumulative density function (ECDF) plot in Figures 5 and 6.

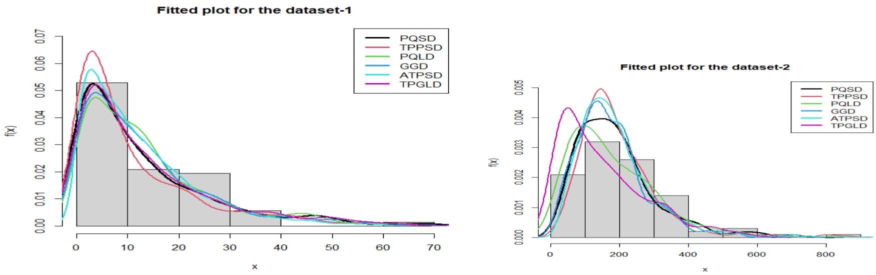

Figure 5. Fitted plot of distributions for the dataset 1 and 2

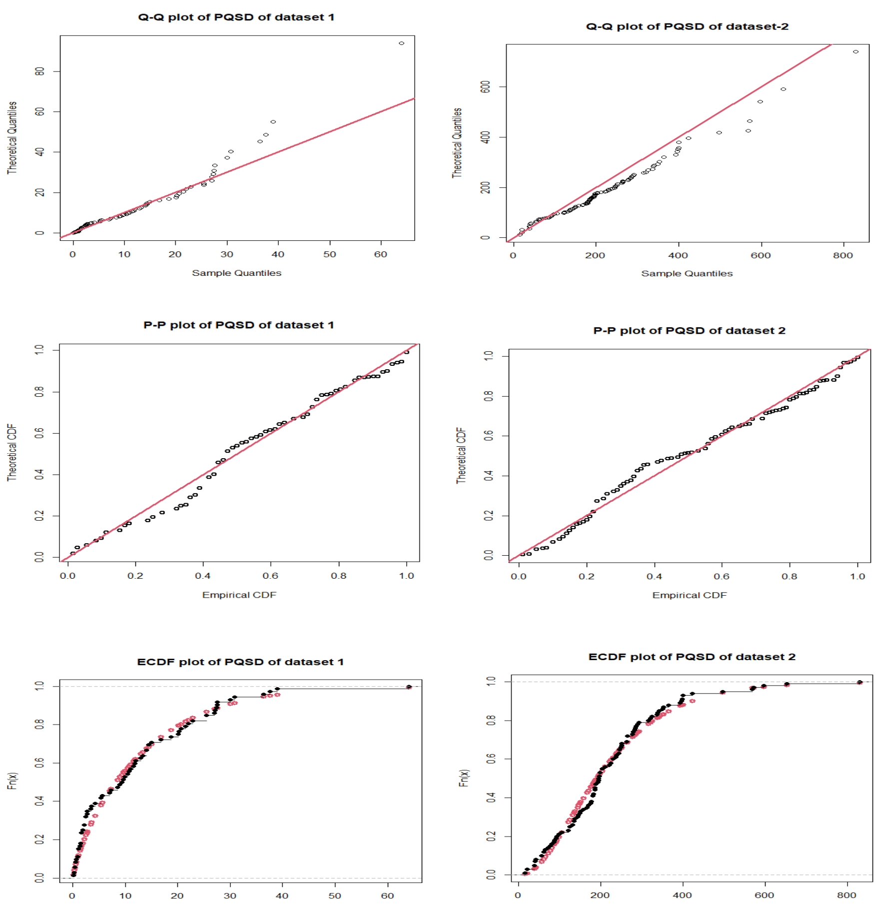

Figure 6. Q-Q plot, P-P plot and ECDF plot of PQSD distribution for the dataset 1 and 2

8. Conclusion

In this paper, PQSD has been proposed. Its moments and statistical properties, such as the survival function, hazard function, reverse hazard function, mean residual life function and stochastic ordering have been analyzed. Parameters of the distribution have been estimated using maximum likelihood estimation and maximum product spacing estimation. To assess the efficiency of the maximum likelihood estimates of the parameters, a simulation study has been presented. The confidence interval of the parameters has been obtained using the Bootstrap confidence interval method. Moreover, applications have been explored for two real lifetime datasets with unimodality and over-dispersion, and the goodness of fit of PQSD has been compared with TPPSD, PQLD, GGD, ATPSD and TPGLD. It has been found that PQSD provides a better fit than TPPSD, PQLD, GGD, ATPSD and TPGLD. The future directions include exploring Bayesian estimation method for the proposed distribution under different loss function, developing the multivariate and the discrete counterpart of the proposed distribution and extending the distribution to deal with the quality control of the product and study regression modelling.

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References

Aderoju, S., Adeniyi, I., (2022). On Power Generalized Akash Distribution with Properties and Applications. *Journal of Statistical Modeling and Analytics*, Vol. 4(1), pp. 1–13.

Alkarni S., (2015). Power quasi-Lindley distribution. Properties and application. *Asian Journal of Mathematics and Computer Research*, Vol. 10(2), pp. 179–195.

Choulakian, V., Stephens, M. A., (2001). Goodness-of-Fit tests for the generalized Pareto distribution. *Technometrices*, Vol. 43, pp. 478–484.

Davison, A. C., Hinkley, D. V. (1997). *Bootstrap methods and their application*. Cambridge University Press.

Efron, B., Tibshirani, R. J. (1993). *An Introduction to the Bootstrap*. Chapman & Hall/ CRC.

Ghitany, M.E., Atieh, B. and Nadarajah, S., (2008). Lindley distribution and its applications. *Mathematics and Computer in Simulation*, Vol. 78, pp. 493–506.

Ghitany, M. E., Al-Mutairi, D. K., Balakrishnan N. and Al-Enezi, L. J., (2013). Power Lindley distribution and associated inference. *Computational Statistics* & *Data Analysis*, Vol. 64, pp. 20–33.

Lindley, D. V., (1958). Fiducial distributions and Bayes' theorem. *Journal of the Royal Statistical Society*, Series B, Vol. 20(1), pp. 102–107.

Murthy, D. N. P, Xie, M. and Jiang, R., (2004), *Weibull models*, John Wily & Sons Inc., Hoboken.

Mussie, T., Shanker, R., (2018). A two parameter Sujatha distribution. *Biometrics* & *Biostatistics International Journal*, Vol. 7(3), pp. 188–197.

Nosakhare, E. and Festus, O., (2018). A three-parameter generalized Lindley distribution: Properties and Application. *Statistica*, Vol. 78(3), pp. 1–17.

Nwikpe, B. J., Iwok, I. A., (2020). A three-parameter Sujatha distribution with application. *European Journal of Statistics and Probability*, Vol. 8(3), pp. 22–34.

Picciotto R., (1970). *Tensile fatigue characteristics of a sized polyester/viscose yarn and their effect on weaving performance*, Master thesis, University of Raleigh, North Carolina State, USA.

Prodhani, H. R., Shanker, R., (2023). On some statistical properties and applications of three-parameter Sujatha distribution. Reliability: *Theory* & *Applications*, Vol. 18(3), pp. 514–527.

Prodhani, H. R., Shanker, R., (2024). A three-parameter power Sujatha distribution with properties and application. *International Journal of Statistics and Reliability Engineering*, Vol. 11(1), pp. 70–78.

Prodhani, H. R., Shanker, R., (2024). Power Pratibha distribution with properties and applications. *International Journal of Statistics and Reliability Engineering*, Vol. 11(2), pp. 217–226.

Shanker, R., Mishra, A., (2013). A quasi-Lindley distribution. African *Journal of Mathematics and Computer Science Research*, Vol. 6(4), pp. 64–71.

Shanker, R., (2016a). Sujatha distribution and its applications. *Statistics in Transition New Series*, Vol. 17(3), pp. 391–410.

Shanker, R., (2016b). A Quasi Sujatha distribution. *International Journal of Probability and Statistics,* Vol. 5(4), pp. 89–100.

Shanker, R., (2016c). Aradhana Distribution and Its applications. *International Journal of Statistics and Applications*, Vol. 6(1), pp. 23–34.

Shanker, R., Shukla, K. K., (2018). A two-parameter power Aradhana distribution with properties and application. *Indian Journal of Industrial and Applied Mathematics*, Vol. 9(2), pp. 210–220.

Shanker, R., Shukla, K. K and Sigh, A. P., (2018). A generalized Akash distribution. *Biometrics* & *Biostatistics International Journal*, Vol. 7(1), pp. 18–26.

Shanker, R., Shukla, K. K., (2019). A two-parameter power Sujatha distribution with properties and application. *International Journal of Mathematics and Statistics*, Vol. 20(3), pp. 11–22.

Shanker, R., (2023). Pratibha distribution with properties and application. *Biometrics* & *Biostatistics International Journal*, Vol. 12(5), pp. 136–142.

Stacy, E. W., (1962). A generalized gamma distribution. *Annals of Mathematical Statistics*, Vol. 33, pp. 1187–1192.

Shukla, K. K., (2018). Pranav distribution with properties and its applications. *Biometrics* & *Biostatistics International Journal*, Vol. 7(3), pp. 244–254.

Shukla, K. K., (2019). Power Pranav distribution and its applications to model lifetime data. *Journal of applied quantitative methods*, Vol. 14(2), pp. 1–13.

Shaked, M., Shanthikumar, J., (1994). *Stochastic Orders and Their Applications*. Academic Press, New York.